Compounding Commuting Matrices

Abstract

We formulate a method of constructing families of commuting matrices of increasing order by sequential compounding. Formulas are derived for sequentially constructing their Jordan forms and singular-value decompositions. Examples are given to illustrate our methods including construction of commuting Latin squares.

Introduction. Although matrices (as arrays) existed much earlier, matrix algebra originated with Arthur Cayley [1] in 1858 and was developed further by many others. As noted by Cayley, in general two matrices $A$ and $B$ do not commute, i.e. $AB \neq BA$. However, certain special families of matrices do commute, e.g. circulant matrices. Other families of commuting matrices are constructed in the present paper together with their Jordan form and singular-value decomposition. We utilize the method of sequential compounding which has a long history as discussed by Rogers, et.al. [6] and used by them in constructions of magic squares and Latin squares. The Kronecker product, denoted by $\otimes$, and its associated identities (see Meyer [4]) is employed in our construction. All matrices in the present paper are square.

Definition. The matrix $A$ is said to be commutal if its elements $a_{ij}$ can be replaced by new variables $a'_{ij}$ to form a matrix $A'$ of the same structural form as $A$ such that $A'$ commutes with $A$. For example, circulant matrices are commutal, as this order-2 one illustrates:

$$A_2 = \begin{bmatrix} a_0 & a_1 \\ a_1 & a_0 \end{bmatrix}, \quad A'_2 = \begin{bmatrix} a'_0 & a'_1 \\ a'_1 & a'_0 \end{bmatrix}, \quad A_2A'_2 = A'_2A_2. \quad (1)$$

Note that $A_2$ is a Latin squares ($a_0$ and $a_1$ appear in all its rows and columns).

Compounding. We compound members of two families of commutal matrices to form a third family of commutal matrices according to the following theorem:

Theorem 1 Let $A \in \mathbb{R}^{m \times m}$ be a commutal matrix and let $B \in \mathbb{R}^{n \times n}$ be a commutal matrix. Let

$$C = A \otimes B, \quad C' = A' \otimes B'.$$

Then $C \in \mathbb{R}^{mn \times mn}$ is a commutal matrix.

Proof. From (2) we have

$$CC' = (A \otimes B)(A' \otimes B') = (AA') \otimes (BB')$$

$$= (A' A) \otimes (B' B) = (A' \otimes B')(A \otimes B) = C'C,$$

whence $CC' = C'C$, i.e. $C$ is commutal. □

For example, compounding of two order-2 circulant matrices (1) gives the following commutal order-4 matrix:

$$C_4 = \begin{bmatrix} a_0 & a_1 \\ a_1 & a_0 \end{bmatrix} \otimes \begin{bmatrix} b_0 & b_1 \\ b_1 & b_0 \end{bmatrix} = \begin{bmatrix} a_0b_0 & a_0b_1 & a_1b_0 & a_1b_1 \\ a_0b_1 & a_0b_0 & a_1b_1 & a_1b_0 \\ a_1b_0 & a_1b_1 & a_0b_0 & a_0b_1 \\ a_1b_1 & a_1b_0 & a_0b_1 & a_0b_0 \end{bmatrix} \quad (4)$$
the elements of which can be relabeled as
\[
C_4 = \begin{bmatrix}
c_0 & c_1 & c_2 & c_4 \\
c_1 & c_0 & c_4 & c_2 \\
c_2 & c_4 & c_0 & c_1 \\
c_4 & c_2 & c_1 & c_0 \\
\end{bmatrix},
\]
(5)

It can be verified directly that \(C_4\) commutes with \(C_4^\dagger\). It is noteworthy that \(C_4\) is a symmetric Latin square. Also, \(C_4\) is a representation of the Klein four-group (Vierergruppe) given by Felix Klein [2] in 1884.

**Jordan Form.** The Jordan form of a matrix is of wide interest as discussed in many textbooks, e.g. Meyer [4]. It can be formed from the eigenvalues and eigenvectors of the matrix (when they exist). Here, the families of commutal matrices with members \(A_i \in \mathbb{R}^{m \times m}\) and \(B_i \in \mathbb{R}^{n \times n}\) are restricted to be diagonalizable with Jordan forms
\[
A_i = S_A D_A S_A^{-1}, \quad B_i = S_B D_B S_B^{-1},
\]
(6)

where \(D_{A_i} \in \mathbb{C}^{m \times m}\) and \(D_{B_i} \in \mathbb{C}^{n \times n}\) are diagonal matrices and the columns of the matrices \(S_A \in \mathbb{C}^{m \times m}\) and \(S_B \in \mathbb{C}^{n \times n}\) are their corresponding eigenvectors. Since \(A_i\) and \(B_i\) are diagonal and members of commuting families, it is known [4] that \(S_A\) and \(S_B\) are the same for all members of each family but \(D_{A_i}\) and \(D_{B_i}\) are different for each member of the family.

The Jordan form of the compounded commutal matrix \(C_i\) of (2) is given by the following theorem:

**Theorem 2** The compounded commutal matrix \(C_i = A_i \otimes B_i\) has the Jordan form
\[
C_i = S_C D_{C_i} S_C^{-1},
\]
(7)

where
\[
S_C = S_A \otimes S_B, \quad D_{C_i} = D_{A_i} \otimes D_{B_i} = S_C^{-1} C_i S_C.
\]
(8)

**Proof.** Using (6), we have
\[
C_i = A_i \otimes B_i = (S_A D_A S_A^{-1}) \otimes (S_B D_B S_B^{-1})
= (S_A \otimes S_B) (D_{A_i} \otimes D_{B_i}) (S_A^{-1} \otimes S_B^{-1}),
\]
(9)
\[
S_C^{-1} = (S_A \otimes S_B)^{-1} = S_A^{-1} \otimes S_B^{-1}.
\]

\(D_{C_i}\) is diagonal since \(D_{A_i}\) and \(D_{B_i}\) are. The relation \(D_{C_i} = S_C^{-1} C_i S_C\) follows from (7).

Furthermore, if \(A_i\) and \(B_i\) are normal matrices \((A_i A_i^T = A_i^T A_i, B_i B_i^T = B_i^T B_i)\), it can be shown that \(C_i\) is normal and [4]
\[
S_{A_i}^{-1} = S_A^*, \quad S_{B_i}^{-1} = S_B^*, \quad S_C^{-1} = S_C^*,
\]
(10)
where \(S^*\) denotes the conjugate transpose of \(S\). In addition, if \(A_i\) and \(B_i\) are symmetric matrices (always normal), then [4]
\[
S_{A_i}^{-1} = S_A^T, \quad S_{B_i}^{-1} = S_B^T, \quad S_C^{-1} = S_C^T,
\]
(11)
i.e. they are orthogonal and \(C_i\) is symmetric too. When \(A_i\) and \(B_i\) are real and symmetric, all \(S\)'s and \(D\)'s are real too. When \(A_i\) and \(B_i\) also are positive definite or positive semi-definite, so is \(C_i\).

For example, in the Jordan form for \(A_2\) of (1),
\[
S_A = S_B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D_A = \text{diag} (a_0 \pm a_1), \quad D_B = \text{diag} (b_0 \pm b_1),
\]
(12)
and in the Jordan form for \( C_4 \) of (5), by (8),

\[
S_C = S_A \otimes S_B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},
\]

\[
D_C = D_A \otimes D_B = S_C^{-1} C_4 S_C = \text{diag} [\lambda_1, \lambda_2, \lambda_3, \lambda_4],
\]

where

\[
\lambda_1 = c_0 + c_1 + c_2 + c_4, \quad \lambda_2 = c_0 - c_1 + c_2 - c_4,
\]

\[
\lambda_3 = c_0 + c_1 - c_2 - c_4, \quad \lambda_4 = c_0 - c_1 - c_2 + c_4.
\]

The Jordan form of \( C_4 \) can be verified directly from (13). Other examples of the Jordan form construction are given in what follows.

**SVD.** The singular-value decomposition (SVD) of a matrix also is of general interest. It has many applications as discussed by Martin and Porter [3] and Meyer [4], where additional references are given. We first review the known facts concerning the SVD and then we consider the SVD for compounded commutal matrices. The SVD of any matrix \( A \in \mathbb{R}^{m \times m} \) reads

\[
A = U \Sigma V^T,
\]

(14)

where \( U \in \mathbb{R}^{m \times m} \) and \( V \in \mathbb{R}^{m \times m} \), are orthogonal matrices \( (U U^T = V V^T = I) \) and \( \Sigma \in \mathbb{R}^{m \times m} \) is a diagonal matrix with \( r = \text{rank} (A) \) positive diagonal elements \( \sigma_k \) (called the singular values) and \( m - r \) zero elements. From (14) we have

\[
A A^T = U \Sigma^2 U^T, \quad A^T A = V \Sigma^2 V^T
\]

(15)

from which \( U \) and \( V \) can be determined subject to (14) being satisfied. For example, if \( U \) and \( \Sigma \) are obtained form (15) and \( A \) is nonsingular \( (r = m) \), then \( V \) is given by

\[
V = A^T U \Sigma^{-1}
\]

(16)

which can be shown to be orthogonal. If \( A \) is singular, then other considerations are necessary, see Meyer [4]. The construction of \( V \) will not be mentioned in the examples that follow.

If \( A \) is normal, then by (6) and (10) the Jordan form of \( A \) reads

\[
A = SDS^* \quad \text{and} \quad A A^T = A A^* = S |D|^2 S^*,
\]

(17)

where the (possibly complex) eigenvalues of \( D \) are denoted by \( \lambda_k \). From (15) and (17) we see that the unique eigenvalues of \( A A^T \) are contained in both \( \Sigma \) and \( |D|^2 \), thus

\[
\sigma_k = |\lambda_k|.
\]

(18)

However, when \( S \) is complex it cannot be used for \( U \) (nor \( S^* \) for \( V^T \)). For symmetric (normal) matrices, \( S \) and \( D \) are real and \( S \) is orthogonal, hence \( U = S \) is allowed. If \( A \) is symmetric and positive semi-definite \( \text{, then } \lambda_k \geq 0 \) and the Jacobi form and SVD of \( A \) are identical.

Now, let \( A_i \in \mathbb{R}^{m \times m} \) be a member of a family of normal commutal matrices \( (A_i A_i^T = A_i^T A_i) \).

It follows that \( A_i A_j^T = A_j^T A_i \), for all \( i, j \) and it can be shown that \( A_i A_j^T \) and \( A_j A_j^T \) commute. Therefore, \( U_{A_i} \) in the SVD of \( A_i \) is the same for all \( A_i \) \( (U_{A_i} = U_A) \), hence

\[
A_i A_i^T = U_{A} \Sigma^2_{A_i} U_{A}^T
\]

(19)

and we have the following theorem:

---

1 The SVD also applies to rectangular and complex matrices, see [3, 4].
Theorem 3. For \( A_i \) and \( B_i \) members of families of normal commutal matrices, the compounded commutal matrix \( C_i = A_i \otimes B_i \) has the SVD

\[
C_i = U_C \Sigma_{C_i} V_{C_i}^T,
\]

where

\[
U_C = U_A \otimes U_B, \quad \Sigma_{C_i} = \Sigma_{A_i} \otimes \Sigma_{B_i} = \left( U_A^T A_i A_i^T U_A \right)^{\frac{1}{2}}, \quad V_{C_i} = V_{A_i}^T \otimes V_{B_i}^T.
\]

Proof.

\[
C_i = A_i \otimes B_i = \left( U_A \Sigma_{A_i} V_{A_i}^T \right) \otimes \left( U_B \Sigma_{B_i} V_{B_i}^T \right) = \left( U_A \otimes U_B \right) \left( \Sigma_{A_i} \otimes \Sigma_{B_i} \right) \left( V_{A_i}^T \otimes V_{B_i}^T \right) = U_C \Sigma_{C_i} V_{C_i}^T.
\]

\( \Sigma_{C_i} \) is diagonal with nonzero elements since \( \Sigma_{A_i} \) and \( \Sigma_{B_i} \) are. It can be shown that \( U_C \) and \( V_{C_i} \) are orthogonal. The relation \( \Sigma_{C_i} = \left( U_A^T A_i A_i^T U_A \right)^{\frac{1}{2}} \) follows from (19). \( \blacksquare \)

As already noted, \( C_i \) is normal when \( A_i \) and \( B_i \) are normal and thus its singular values are related to its eigenvalues by a relation of the form (18). From (21) and (8) we have

\[
U_C = \left( U_A S_A^* \right) \otimes \left( U_B S_B^* \right) = \left( U_A S_A^* \right) \otimes \left( U_B S_B^* \right) (S_A \otimes S_B),
\]

i.e. the SVD matrix \( U_C \) is related to the eigenvector matrix \( S_C \) by

\[
U_C = L S_C, \quad L = \left( U_A S_A^* \right) \otimes \left( U_B S_B^* \right) \quad LL^* = I.
\]

Examples of the SVD construction are given in what follows.

Example 1. Here is an order-8 normal commutal matrix constructed from compounding \( C_4 \) of (5) with \( C_2 \) of (1) and relabeling the elements as before:

\[
C_8 = \begin{bmatrix}
c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\
c_1 & c_0 & c_3 & c_2 & c_5 & c_4 & c_7 & c_6 \\
c_2 & c_3 & c_0 & c_1 & c_6 & c_7 & c_4 & c_5 \\
c_3 & c_2 & c_1 & c_0 & c_7 & c_6 & c_5 & c_4 \\
c_4 & c_5 & c_6 & c_7 & c_0 & c_1 & c_2 & c_3 \\
c_5 & c_4 & c_7 & c_6 & c_1 & c_0 & c_3 & c_2 \\
c_6 & c_7 & c_4 & c_5 & c_2 & c_3 & c_0 & c_1 \\
c_7 & c_6 & c_5 & c_4 & c_3 & c_2 & c_1 & c_0
\end{bmatrix}.
\]

Again, it is noteworthy that \( C_8 \) is a symmetric Latin square. In view of its symmetry, \( U_{C_S} = S_{C_S} \) and \( \sigma_i = |\lambda_i| \). Furthermore, repeated application of our compounding construction using (1) produces symmetric Latin squares of order \( 2^k \). Also, commutal normal Latin squares of any nonprime order can be produced by compounding two circulant matrices which are commutal and normal. The Jordan form and SVD of such matrices can be constructed sequentially from our formulas.

Example 2. We construct an order-6 commutal matrix by compounding an order-3 commutal matrix with an order-2 commutal matrix. We start with the symmetric order-3 commutal matrix

\[
A = \begin{bmatrix}
a_0 & a_1 & a_2 \\
a_1 & a_0 + a_2 & a_1 \\
a_2 & a_1 & a_0
\end{bmatrix}.
\]
which commute as required. The Jordan form of \( A \) has

\[
S_A = \frac{1}{2} \begin{bmatrix}
-\sqrt{2} & 1 & -\sqrt{2} \\
0 & \sqrt{2} & 1 \\
\sqrt{2} & 0 & 1
\end{bmatrix}, \quad D_A = \text{diag} \left( a_0 - a_2, a_0 + a_2 \pm \sqrt{2}a_1 \right).
\]  

(26)

From \( D_A \) we see that \( A \) with integer elements is nonsingular (\( \lambda_i \neq 0 \)) unless \( a_0 = a_2 \) or \( a_0 = a_2, a_1 = 0 \). Furthermore, \( A \) is positive semi-definite if \( a_0 \geq a_2 \) and \( a_0 + a_2 \geq \pm \sqrt{2}a_1 \). The SVD \( A \) has

\[
U_A = S_A, \quad \Sigma_A = \text{diag} (\sigma_1, \sigma_2, \sigma_3), \quad \sigma_i = |\lambda_i|.
\]  

(27)

Next, we introduce the following order-2 commutal normal matrix:

\[
B = \begin{bmatrix}
b_0 & b_1 \\
-b_1 & b_0
\end{bmatrix}
\]  

(28)

whose Jordan form has

\[
S_B = \frac{\sqrt{2}}{2} \begin{bmatrix}
1 & 1 \\
-i & i
\end{bmatrix}, \quad [S_B]^* = \frac{\sqrt{2}}{2} \begin{bmatrix}
1 & i \\
1 & -i
\end{bmatrix}, \quad D_B = \text{diag} (b_0 \mp ib_1).
\]  

(29)

From (19), the SVD of \( B \) has

\[
U_B = V_B = I_2, \quad \Sigma_B = \text{diag} (\sigma_1, \sigma_2), \quad \sigma_1 = \sigma_2 = (b_0^2 + b_1^2)^{1/2} = |\lambda_1| = |\lambda_2|.
\]  

(30)

On compounding \( A \otimes B \) we obtain the order-6 commutal matrix \( C \) which after relabeling its elements reads

\[
C = \begin{bmatrix}
c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\
-c_1 & c_0 & -c_3 & c_2 & -c_5 & c_4 \\
c_2 & c_3 & c_0 + c_4 & c_1 + c_5 & c_2 & c_3 \\
-c_3 & c_2 & -c_1 - c_5 & c_0 + c_4 & -c_3 & c_2 \\
c_4 & c_5 & c_2 & c_3 & c_0 & c_1 \\
-c_5 & c_4 & -c_3 & c_2 & -c_1 & c_0
\end{bmatrix}.
\]  

(31)

By (8), the Jordan form of \( C \) has

\[
S_C = \frac{1}{4} \begin{bmatrix}
-2 & -2 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
2i & -2i & -i\sqrt{2} & i\sqrt{2} & -i\sqrt{2} & i\sqrt{2} \\
0 & 0 & 2 & 2 & -2 & -2 \\
0 & 0 & -2i & 2i & 2i & -2i \\
2 & 2 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
-2i & 2i & -i\sqrt{2} & i\sqrt{2} & -i\sqrt{2} & i\sqrt{2}
\end{bmatrix}, \quad S_C^{-1} = S_C^T,
\]

\[
D_C = \text{diag} (\lambda_1, \lambda_2, \ldots, \lambda_6),
\]  

(32)

where

\[
\lambda_1 = c_0 - c_4 - i(c_1 + c_3), \quad \lambda_2 = \bar{\lambda}_1, \\
\lambda_3 = c_0 + c_4 + \sqrt{2}c_2 - i\left(c_1 + c_5 + \sqrt{2}c_3\right), \quad \lambda_4 = \bar{\lambda}_3, \\
\lambda_5 = c_0 + c_4 - \sqrt{2}c_2 - i\left(c_1 + c_5 - \sqrt{2}c_3\right), \quad \lambda_6 = \bar{\lambda}_5.
\]
as can be verified directly. From $\mathbf{D}_C$, we see that $\mathbf{C}$ with integer elements is nonsingular unless $c_0 = c_4$, $c_1 = c_5$ or $c_0 = -c_4$, $c_1 = -c_5$, $c_2 = c_3 = 0$. By (21) with (27) and (20), the SVD of $\mathbf{C}$ has

$$
\mathbf{U}_C = \mathbf{S} \otimes \mathbf{I}_2 = \frac{1}{2} \begin{bmatrix}
-\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 \\
0 & 1 & 0 & 0 & -\sqrt{2} & 0 \\
0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{2} \\
0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{2} \\
\sqrt{2} & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} \\
0 & \sqrt{2} & 0 & \sqrt{2} & 0 & \sqrt{2}
\end{bmatrix}.
$$

(33)

By (19) with (33), the singular values of $\hat{A}_6$ are

$$
\Sigma_C = \left[\mathbf{U}_C^T \mathbf{A}_C \mathbf{A}_C^T \mathbf{U}_C\right]^{\frac{1}{2}} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_6), \quad \sigma_k = |\lambda_k|, \quad k = 1, 2, \ldots, 6.
$$

(34)
as can be verified directly. The relation between $\sigma_k$ and $\lambda_k$ follows from (18) since $\mathbf{C}$ is a normal matrix (because $\mathbf{A}$ and $\mathbf{B}$ are normal). However, $\mathbf{S}_C$ is complex and therefore it cannot be used as $\mathbf{U}_C$ instead of (33). The relation (23) between $\mathbf{S}_C$ and $\mathbf{U}_C$ has

$$
\mathbf{L}_C = (\mathbf{U}_A \mathbf{S}_A^*) \otimes (\mathbf{U}_B \mathbf{S}_B^*) = \mathbf{I}_3 \otimes \mathbf{S}_B^* = \frac{\sqrt{2}}{2} \begin{bmatrix}
1 & i & 0 & 0 & 0 & 0 \\
1 & -i & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & i & 0 & 0 \\
0 & 0 & 1 & -i & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & i \\
0 & 0 & 0 & 0 & 1 & -i
\end{bmatrix}.
$$

(35)

There are many other possibilities for constructing an order-6 commutal matrix using various combinations of (25), (1), (5), and an order-3 circulant matrix. One could start with an order-2 matrix and use two order-3 commutal matrices in the construction. There are endless possibilities for constructing higher-order commutal matrices sequentially by repeated compounding as the reader should enjoy doing.

References


