Comparison of Jacobi and Gauss-Seidel Iterative Methods for the Solution of Systems of Linear Equations

ABSTRACT

In this research work two iterative methods of solving system of linear equation has been compared, the iterative methods are used for solving sparse and dense system of linear equation and the methods were being considered are: Jacobi method and gauss-seidel method. The results show that gauss-seidel method is more efficient than Jacobi method by considering maximum number of iteration required to converge and accuracy.

Keywords: Iterative methods; Linear equations problem; Convergence; square matrix.

1. INTRODUCTION

The development of numerical methods on a daily basis is to find the right solution techniques for solving problems in the field of applied science and pure science, such as weather forecasts, population, the spread of the disease, chemical reactions, physics, optics and others.

Many problems (Kendall E 2007) in applied mathematics involve solving systems of linear equations, with the linear system occurring naturally in some cases and as a part of the solution process in other cases

Collections of linear equations are called linear systems of equations. They involve same set of variables. Various methods have been introduced to solve systems of linear equations (N. A. Saeed and A.Bhatti 2008). There is no single method that is best for all situations. These methods should be determined according to speed and accuracy. Speed is an important factor in solving large systems of equations because the volume of computations involved is huge.(Noreen Jamil 2015)

Systems of linear equations (Kalambi 2008) arise in a large number of areas both directly in modeling physical situations and indirectly in the numerical solutions of the other mathematical models. These applications occur in virtually all areas of the physical, biological and social science. A linear equation in the variable $x_1,\ x_2, . . ., x_n$ is any equation of the form $a_1x_1 + a_2x_2 + . . . + a_nx_n = b_1$ (1)

It is fast and simple to use when the coefficient matrix is sparse. There are many approaches of solving system of linear equations i.e. direct methods and Indirect (iterative) methods. The direct methods give the exact solution in which there is no error except the round off error due to the machine, whereas iterative methods give the approximate solutions in which there is some error (lascar A. H. and Samira Behera, 2014).

2. JACOBI METHOD

The first iterative technique is called the Jacobi method, after Carl Gustav Jacob Jacobi (1804–1851), a German mathematician who was one of the famous algorists formulate an iterative method of solving system of linear equations. This method makes two assumptions:

(1) That the system given by has a unique solution and

(2) That the coefficient matrix A has no zeros on its main diagonal.

If any of the diagonal entries are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal.

3. GAUSS-SIEDEL METHOD

The next method is called Gauss-Seidel method, which the modification of Jacobi method, named after Carl Friedrich Gauss (1777–1855) and Philipp L. Seidel (1821–1896). This modification is no more difficult to use than the Jacobi method, and it often requires fewer iterations to produce the same degree of accuracy. With the Jacobi method, the values of obtained in the nth approximation remain unchanged until the entire nth approximation has been calculated. With the Gauss- Seidel method, on the other hand, we use the new values of each as soon as they are known. That is, once we have determined from the first equation, its value is then used in the second equation to obtain the new values. Similarly, the new value and the first value are used in the third equation to obtain the new and so on

4. ANALYSIS METHOD 1

The Jacobi method was obtain by solving the ith equation in Ax = b, to obtain x_i

(Provided $a_{ii} \neq 0$) i. e. given a system of linear equation

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = b_3 \\
. \quad . \quad . \quad . \quad . \quad . \\
. \quad . \quad . \quad . \quad . \quad . \\
. \quad . \quad . \quad . \quad . \quad . \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n
\end{array} \tag{2}
$$

To begin, solve first equation for x_1, second equation for x_2 , third equation for x_3 and so on to obtain

$$
\begin{array}{l}
x_1^{k+1} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^k - a_{13}x_3^k - \ldots - a_{1n}x_n^k\right) \\
x_2^{k+1} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^k - a_{23}x_3^k - \ldots - a_{2n}x_n^k\right) \\
x_3^{k+1} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^k - a_{32}x_2^k - \ldots - a_{3n}x_n^k\right) \\
\qquad \vdots \\
x_n^{k+1} = \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1^k - a_{n2}x_2^k - \ldots - a_{nn-1}x_{n-1}^k\right)
\end{array} \tag{3}
$$

Then make initial guess (zero iteration) for the solution $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)} \ldots x_n^{(0)})$ substitute these value into the right hand side of (3). This constitute first iteration

$x^1 = (x_1^{(1)}, x_2^{(1)}, x_3^{(1)} \ . \ . \ . x_n^{(1)})$. Second iteration is obtained by substituting first iteration into

the left hand side of (3.2), that is $x^2 = (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, . \ . \ . \ x_n^{(2)})$. And so on. The Jacobi method can be generalize as for each k ≥ 0 we can generate the component x_i^{k+1} of x^{k+1} from x^k by

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1 \\ j\neq i}}^{n}\left(-a_{ij}x_j^k\right) + b1\right] \qquad \text{For } i = 1, 2 \ldots n$$

The Jacobi method in matrix form can be found by considering an n×n system of linear equation Ax = b where

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}, \ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}. \text{ We split matrix}$$

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & 0 & 0 & \cdots & a_{nn} \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ -a_{21} & 0 & \cdots & 0 & 0 \\ -a_{31} & -a_{32} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{n,n-1} & 0 \end{pmatrix} -$$

$$\begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ 0 & -a_{22} & -a_{23} & \cdots & -a_{2n} \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -a_{n-1,n} \\ O & 0 & 0 & \cdots & 0 \end{pmatrix}. \qquad = D - L - U.$$

Therefore the matrix Ax =b can be transformed into (D – L – U) x = b, this implies that

Dx = (L +U)x + b.

5. ANALYSIS OF METHOD 2

With Jacobi method, the value of x^{k+1} was obtain in (k+1)th iteration remain unchanged until the entire (k+1)th iteration has been calculated. With gauss-seidel method we use the value of x_i^{k+1} as soon as they are known. That is

$$
\begin{aligned}
x_1^{k+1} &= \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^k - a_{13}x_3^k - \ldots - a_{1n}x_n^k\right)\\
x_2^{k+1} &= \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{k+1} - a_{23}x_3^k - \ldots - a_{2n}x_n^k\right)\\
x_3^{k+1} &= \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{k+1} - a_{32}x_2^{k+1} - \ldots - a_{3n}x_n^k\right)\\
&\vdots\\
x_n^{k+1} &= \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1^{k+1} - a_{n2}x_2^{k+1} - \ldots - a_{nn-1}x_{n-1}^{k+1}\right)
\end{aligned}
\qquad (4)
$$

Then make initial guess (zero iteration) for the solution $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)} \ldots x_n^{(0)})$ substitute these value into the righ t hand side of (3.3). This constitute first iteration, $x^1 = (x_1^{(1)}, x_2^{(1)}, x_3^{(1)} \ldots x_n^{(1)})$. Second iteration is obtained by substituting first iteration into the left hand side of (3.3), $x^2 = (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots x_n^{(2)})$. And so on. This method can be generalize as for each k ≥ 0 we can generate the component x_i^{k+1} of x^{k+1} from x^k by

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i=1}\left(a_{ij}x_j^k\right) - \sum_{j=i+1}^{n}\left(a_{ij}x_j^{k+1}\right) + b1\right] \qquad \text{For } i = 1, 2 \ldots n$$

The gauss-seidel method in a matrix form is given by;

$(D - L)x^{k+1} = Ux^{k} + b$

$x^{k+1} = (D - L)^{-1}Ux^{k} + (D - L)^{-1}b$

6. CONVERGENCE OF ITERATIVE METHOD

The rate of convergence of iterative methods determine how fast the error $|x^{k} - x|$ goes to zero as k, the number of iteration increases. The sufficient condition of convergence for iterative methods define as $x^{k+1} = Bx^{K} + c$ to converge is that ρ(B) = $\max_{1<i<n} |\lambda_i(B)| < 1$ where ρ (B) is the spectral radius of B. This condition is fulfilled for both Jacobi and Gauss-seidel methods if the coefficient matrix is diagonally dominant for any choice of initial approximation.

THEOREM (6.1): For any x^{o} in R^{n} the sequence $\{x^{k+1}\}_{k=0}^{\infty}$ define by $x^{K+1} = Tx^{k} + c$, for each k≥ 0 converge to a unique solution of x = Tx + c iff ρ(T) < 1

6.2 **Absolute error** =|True value – approximate value|

6.3 **Relative error:** = $\frac{Absolute\ error}{|True\ value|}$

6.4 **percentage relative error** =Relative error × 100%

7. NUMERICAL EXPERIMENTS

The problem in system of linear equations shall be considered

Problem 7.1

Solve the equation using Jacobi's method.

$4x_1 - x_2 - x_4 = 0$

$-x_1 + 4x_2 - x_3 - x_5 = 5$

$-x_2 + 4x_3 - x_6 = 0$

$-x_1 + x_4 - x_5 = 6$

$-x_2 - x_4 + 4x_5 - x_6 = -2$

$-x_3 - x_5 + 4x_6 = 6$

Taking the initial approximation $x_1^k = x_2^k = x_3^k = x_4^k = x_5^k = x_6^k = 0$ starting with these value and continuous to iterate we obtain the solution in the table below

TABLE 1: Iteration Result for Jacobi Method

Iteration	X_1	X_2	X_3	X_4	X_5	X_6
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	1.250000	0.000000	1.500000	-0.50000	1.500000
2	0.687500	1.125000	0.687500	1.375000	0.562500	1.375000

3	0.625000	1.73445	0.625000	1.812575	0.46875	1.812575
4	0.886756	1.679688	0.886756	1.773438	0.839900	1.773438
5	0.823282	1.898554	0.823282	1.926865	0.806641	1.926865
6	0.956355	1.883301	0.956355	1.917481	0.938668	1.917481
7	0.950196	1.962695	0.950196	1.973606	0.929566	1.973606
8	0.984075	1.957490	0.984075	1.969941	0.977477	1.969941
9	0.981858	1.986497	0.981858	1.990388	0.974343	1.990388
10	0.994199	1.984515	0.994199	1.989050	0.991796	1.989050
11	0.993391	1.995041	0.993391	1.996499	0.990684	1.996499
12	0.997887	1.994354	0.997887	1.996011	0.997011	1.996011
13	0.997593	1.99816	0.997593	1.998725	0.996595	1.998728
14	0.999230	1.997945	0.999230	1.998547	0.998912	1.998547
15	0.999123	1.999343	0.999123	1.999536	0.998760	1.999536
16	0.999720	1.999252	0.999720	1.999471	0.999604	1.999471
17	0.999681	1.999761	0.999681	1.999831	0.999549	1.999831
18	0.999898	1.999728	0.999898	1.999808	0.999856	1.999808

This complete the table of the solution to the system of linear equation given above and the values of x_i are $(x_1, x_2, x_3, x_4, x_5, x_6)$ = (0.999898, 1.999728, 0.999898, 1.999808, 0.999856, 1.999808) respectively.

Problem 7.2

Solve the equation using gauss-seidel method.

$4x_1 - x_2 - x_4 = 0$

$-x_1 + 4x_2 - x_3 - x_5 = 5$

$-x_2 + 4x_3 - x_6 = 0$

$-x_1 + x_4 - x_5 = 6$

$-x_2 - x_4 + 4x_5 - x_6 = -2$

$-x_3 - x_5 + 4x_6 = 6$

Taking the initial approximation $x_1^k = x_2^k = x_3^k = x_4^k = x_5^k = x_6^k = 0$

TABLE 2: : Iteration Result for Gauss-Seidel Method

Iteration	X_1	X_2	X_3	X_4	X_5	X_6
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	1.250000	0.312500	1.500000	0.187500	1.625000
2	o.687500	1.546875	0.79296	1.718750	0.722656	1.878906
3	0.816406	1.833008	0.927979	1.884766	0.899188	1.956792
4	0.929444	1.939153	0.973986	1.957158	0.963276	1.984316
5	0.974078	1.977835	0.990538	1.984339	0.986623	1.994290

6	0.990544	1.991926	0.996554	1.994292	0.995127	1.997920
7	0.996555	1.997059	0.998745	1.997921	0.998225	1.999243
8	0.998745	1.998929	0.999543	1.999243	0.999354	1.999724
9	0.999543	1.999610	0.999834	1.999724	0.999765	1.999900
10	0.999834	1.999858	0.999940	1.999900	0.999915	1.999964

Hence the solution is obtain after ten successive iteration we have x_1 =0.999834, $x_2 =$ 1.999858, x_3 =0.999940, x_4 =1.999900, x_5 =0.999915, x_6 =1.999964

7.2 Error analysis of Jacobi method

The true value are $(x_1, x_2, x_3, x_4, x_5, x_6)$ = (1, 2, 1, 2, 1, 2) while the computed value are (0.999898, 1.999728, 0.999898, 1.999808, 0.999856, 1.999808). Hence we determine the error as follows by using x_2

Absolute error =|True value – approximate value| =|2 - 1.999728|=0.000272

Percentage relative error = $\frac{Absolute\ error}{|True\ value|}$ = 0.000136*100% =0.0136%

7.3 Error analysis of gauss-seidel method

The true value are $(x_1, x_2, x_3, x_4, x_5, x_6)$ = (1, 2, 1, 2, 1, 2) while the computed value are, (0.999834, 1.999858, 0.999940, 1.999900, 0.999915, 1.999964). Hence we determine the error as follows by using x_2

Absolute error =|True value – approximate value| =|2 - 1.999834|=0.000166

Percentage relative error = $\frac{Absolute\ error}{|True\ value|}$ = 0.000166*100% =0.0083%

7.4 COMPARISON OF TWO ITERATIVE METHODS USED IN THE EVALUATION

TABLE 3 showing percentage error of both methods

METHODS	NO OF ITERATION	ERROR%
JACOBI METHOD	18	0.0136
GAUSS-SIEDAL METHOD	10	0.0083

8 Conclusion

There are different methods of solving system of linear equation; some are direct methods while some are numerical method (iterative method). In this research work, two iterative methods of solving system of linear equations have been presented where the gauss-seidel method proved to be the best and effective in the sense that it converges very fast. From the practical example, we observe that the required solution was obtained with very little iteration without much problem relating to the starting condition.

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