Data Article

Numerical Solution of Volterra- Fredholm integral equations using Hybrid Orthonormal Bernstein and Block-Pulse Functions

Abstract

we have proposed an efficient numerical method to solve a class of mixed Volterra-Fredholm integral equations (VFIE’s) of the second kind, numerically based on Hybrid Orthonormal Bernstein and Block-Pulse Functions (OBH). The aim of this paper is to apply OBH method to obtain approximate solutions of nonlinear Fuzzy Fredholm Integro-differential Equations. First we introduce properties of Hybrid Orthonormal Bernstein and Block-Pulse Functions, we used it to transform the integral equations to the system of linear algebraic equations then an iterative approach is proposed to obtain approximate solution of class of linear algebraic equations, a numerical examples is presented to illustrate the proposed method. The error estimates of the proposed method is given.

Keywords: Hybrid orthonormal Bernstein and Block-Pulse Functions, linear Volterra-Fredholm integral equations, Integration of the cross product, Product matrix, Coefficient matrix.

1. Introduction

Integral equations are encountered in various fields of science and numerous applications such as physics [1], biology [2] and engineering [3,4]. But we can also use it in numerous applications, such as biomechanics, control, economics, elasticity, electrical engineering, electrodynamics, electrostatics, filtration theory, fluid dynamics, game theory, heat and mass transfer, medicine, oscillation theory, plasticity, queuing theory, etc. [5]. Fredholm and Volterra integral equations of the second kind show up in studies that includes airfoil theory [6], elastic contact problems [7,8], fracture mechanics [9], combined infrared radiation and molecular conduction [10] and so on.

Numerical Solution Of Linear Volterra-Fredholm Integral Equations, such as Block-Pulse functions [14 - 19], Triangular functions [20 - 22], Haar functions [23], Hybrid Legendre and Block-Pulse functions [24 - 25], Hybrid Chebyshev and Block-Pulse functions [25- 26], Hybrid Taylor, Block-Pulse functions [27], Hybrid Fourier and Block-Pulse functions In recent years, many researchers have been successfully applying Bernstein polynomials method (BPM) to various linear and nonlinear integral equations . For example, Bernstein polynomials method is applied to find an approximate solution for Fredholm integro-Differential equation and integral equation of the second kind in (AL-Juburee 2010). (Al-A'asam 2014) used Bernstein polynomials for deriving the modified Simpson's 3/8 ,and the composite modified Simpson's 3/8 to solve one dimensional linear Volterra integral equations of the second kind. Application of two-dimensional Bernstein polynomials for solving mixed Volterra-Fredholm integral equations can be found in(Hosseini.et al 2014) . In this paper, Hybrid Orthonormal Bernstein and Block-Pulse Functions (OBH) to solve mixed Volterra-Fredholm integral equations (VFIE's) of the second kind:

$$u(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)\, u(t)\, dt + \lambda_2 \int_a^b k_2(x,t)\, u(t)\, dt$$

where $a \le x \le b, \lambda_1, \lambda_2$ are scalar parameters, $f(x), k_1(x,t), k_2(x,t)$ are continuous functions and $u(x)$ is the unknown function to be determine.

The advantage of this method to other existing methods is its simplicity of implementation besides some other advantages.

This paper is organized as follows: In Section 2, we introduce Bernstein polynomials and their properties. Also we orthonormal these polynomials and hybrid them with Block-Pulse functions to obtain new basis. In Section 3, these new basis together with collocation method are used to reduce the linear Volterra-fredholm integral equation to a linear system that can be solved by various method. Section 4 illustrates some applied models to show the convergence, accuracy and advantage of the proposed method and compares it with some other existed method. In Section 5, numerical experiments are conducted to demonstrate the viability and the efficiency of the proposed method computationally. Finally Section 6 concludes the paper.

2. BASIC DEFINITION

In this section we introduce Bernstein polynomials and their properties to get better approximation, we orthonormal these polynomials and hybrid them with Block-Pulse functions.

A. Definition of Bernstein polynomials

B-polynomials (Bernstein polynomials basis) of nth-degree were introduced in the approximation of continuous functions f(x) on an interval [0, 1] (see [11]),

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \qquad 0 \le i \le n. \tag{1}$$

There are (n +1) nth-degree polynomials and for convenience, we set $B_{i,n}(x) = 0$, if $i < 0$ or $i > n$.

A recursive definition also can be used to generate the B-polynomials over this interval, so that the ith nth degree B-polynomial can be written;

$$B_{i,n}(x) = (1-x)B_{i,n-1}(x) + xB_{i-1,n-1}(x) \tag{2}$$

The explicit representation of the orthonormal Bernstein polynomials, denoted by ($OB_{i,n}(x)$) here, was discovered by analyzing the resulting orthonormal polynomials after applying the Gram-Schmidt process on sets of Bernstein polynomials of varying degree n. For example, for $n = 5$, using the Gram-Schmidt process on $OB_{i,5}(x)$ normalizing, and simplifying the resulting functions, we get the following set of orthonormal polynomials;

$$OB_{0,5}(x) = \sqrt{11}\,(1-t)^5$$

$$OB_{1,5}(x) = 3(1-t)^4(11t-1)$$

$$OB_{2,5}(x) = \sqrt{7}\,(1-t)^3(55t^2 - 20t + 1)$$

$$OB_{3,5}(x) = \sqrt{5}\,(1-t)^2(165t^3 - 135t^2 + 27t - 1)$$

$$OB_{4,5}(x) = \sqrt{3}\,(1-t)(330t^4 - 480t^3 + 216t^2 - 32t + 1)$$

$$OB_{5,5}(x) = (462t^5 - 1050t^4 + 840t^3 - 280t^2 + 35t - 1)$$

We can see from these equations that the orthonormal Bernstein polynomials are, in general, a product of a factorable polynomial and a non-factorable polynomial. For the factorable part of these polynomials, there exists a pattern of the form

$$(\sqrt{2(n-i)+1})(1-t)^{n-i} \qquad i = 0,1,....,n.$$

While it is less clear that there is a pattern in the non-factorable part of these polynomials, the pattern can be determined by analyzing the binomial coefficients present in Pascal's triangle. In doing this, we have determined the explicit representation for the orthonormal Bernstein polynomials to be

$$OB_{i,n}(x) = (\sqrt{2(n-j)+1})(1-t)^{n-i} \sum_{k=0}^{i} (-1)^K \binom{2n+1-k}{i-k} \binom{i}{k} t^{i-K} \tag{3}$$

B. Definition of Block-Pulse functions (BPFs) and their properties

BPFs are studied by many authors and applied for solving different problems, for example see [12].

A k - set of BPFs over the interval $[0, T)$ is defined as

$$B_i(t) = \begin{cases} 1, & \frac{iT}{k} \le t < \frac{(i+1)T}{k} \\ 0, & elsewhere \end{cases}, i = 0,1,....,k-1. \quad (4)$$

with a positive integer value for k . In this paper, it is assumed that $T = 1$, so BPFs are defined over $[0, 1)$. BPFs have some main properties, the most important of these properties are disjointness, orthogonality, and completeness.

(1) The disjointness property can be clearly obtained from the definition of BPFs

$$B_i(t)B_j(t) = \begin{cases} B_i(t), & i = j \\ 0, & i \neq j \end{cases} \qquad i, j = 0,1,....,k-1 \quad (5)$$

(2) The orthogonality property of these functions is

$$\langle B_i(t), B_j(t) \rangle = \int_0^1 B_i(t)\, B_j(t)\, dt = \begin{cases} \frac{1}{k}, & i = j \\ 0, & i \neq j \end{cases} \qquad i, j = 0,1,....,k-1 \quad (6)$$

(3) The third property is completeness. For every $y \in L^2[0,1)$, when k approaches to the infinity, Parseval's identity holds, that is

$$\int_0^1 y^2(t)\, dt = \sum_{i=1}^{\infty} c_i^2 \left\| B_i(t) \right\|^2$$

where $$c_i = k \int_0^1 f(t)\, B_i(t)\, dt \quad (7)$$

3. Some properties of hybrid functions

A. Hybrid functions of block-pulse and Orthonormal Bernstein polynomials

We define OBH on the interval [0; 1] as follow:

$$OBH_{i,j}(x)=\begin{cases} B_{j,n}(M\,x-i+1) & \frac{i-1}{M}\le x<\frac{i}{M} \\ 0 & otherewise \end{cases} \qquad (8)$$

where $i=1,2,....M$ and $j=0,1,2,....n$. thus our new basis is $\{OBH_{1,0},OBH_{1,1},...,OBH_{M,n}\}$ and we can approximate function with this base. for example for M = 2 and n = 1

$$OBH_{1,0}(x)=\begin{cases} (-2x+1) & 0\le x<\frac{1}{2} \\ 0 & otherewise \end{cases}$$

$$OBH_{2,0}(x)=\begin{cases} (2x) & \frac{1}{2}\le x<1 \\ 0 & otherewise \end{cases}$$

$$OBH_{1,1}(x)=\begin{cases} (-2x+2) & 0\le x<\frac{1}{2} \\ 0 & otherewise \end{cases}$$

$$OBH_{2,1}(x)=\begin{cases} (2x-1) & \frac{1}{2}\le x<1 \\ 0 & otherewise \end{cases}$$

B. Function approximation by using OBH functions

Any function $y(t)$ which is square integrable in the interval $[0,1)$ can be expanded in a hybrid Orthonormal Bernstein and Block-Pulse Functions

$$y(t)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}c_{ij}OBH_{ij}(t),\ i=1,2,...,\infty,\ j=0,1,2,...,\infty,t\in[0,1), \qquad (9)$$

where the hybrid Orthonormal Bernstein and Block-Pulse coefficients

$$c_{ij}=\frac{(y(t),OBH_{ij}(t))}{(OBH_{ij}(t),OBH_{ij}(t))} \qquad (10)$$

In (10), $(.,.)$ denotes the inner product. Usually, the series expansion Eq. (9) contains an infinite number of terms for a smooth $y(t)$. If $y(t)$.is piecewise constant or may be approximated as piecewise constant, then the sum in (9) may be terminated after nm terms, that is

$$y(t) \cong \sum_{i=1}^{M} \sum_{j=0}^{n} c_{ij} OBH_{ij}(t) = C^T OBH(t) \tag{11}$$

where

$OBH(x) = [OBH_{1,0}, OBH_{1,1}, ..., OBH_{M,n}]^T$,

and

$C = [c_{1,0}, c_{1,1}, ..., c_{M,n}]^T$

Therefore we have

$C^T < OBH(x), OBH(x) >=< u(x), OBH(x) >$

then

$C = D^{-1} < u(x), OBH(x) >$,

where

$$D =< OBH(x), OBH(x) >,$$

$$= \int_0^1 OBH(x).OBH^T(x) dx \tag{12}$$

$$= \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & D_M \end{pmatrix}$$

then by using (7) $D_i (i = 1, 2, ..., M)$ is defined as follow:

$$(D_n)_{i+1, j+1} = \int_{\frac{i-1}{M}}^{\frac{i}{M}} B_{i,n}(Mx - i + 1) B_{j,n}(Mx - j + 1) dx$$

$$= \frac{1}{M} \int_0^1 B_{i,n}(x) B_{j,n}(x) dx$$

$$= \frac{\binom{n}{i}\binom{n}{j}}{M(2n+1)\binom{2n}{i+j}}$$

We can also approximate the function $k(x,t) \in L[0,1]$ as follow:

$k(x,t) \approx OBH^T(x) K\, OBH(t)$,

where K is an $M(n+1)$ matrix that we can obtain as follows:

$$K = D^{-1} < OBH(x) < k(x,t), OBH(t) >> D^{-1} \quad (13)$$

C. Integration of OBH functions

In OBH function analysis for a dynamic system, all functions need to be transformed into OBH functions. Since the differentiation of OBH functions always results in impulse functions which must be avoided, the integration of OBH functions is preferred. The integration of OBH functions should be expandable into OBH functions with the coefficient matrix P.

$$\int_0^t OBH_{(n\times(m+1))}(\tau)\, d(\tau) \approx P_{n(m+1)\times n(m+1)}\, OBH_{(n\times(m+1))}(t), t \in [0,1), \quad (14)$$

where the $n(m+1)$-square matrix P is called the operational matrix of integration, and $OBH_{(n\times(m+1))}(t)$ is defined in Eq. (8). A subscript $n(m+1)\times n(m+1)$ of P denotes its dimension and P is given in [4] as:

$$\mathrm{P}_{\mathrm{n(m+1)}\times n(m+1)} = \begin{bmatrix} \mathrm{H} & \mathrm{G} & G & \cdots & G \\ 0 & \mathrm{H} & G & \cdots & G \\ 0 & 0 & H & \cdots & G \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & H \end{bmatrix} \quad (15)$$

$$\mathrm{G}_{\mathrm{n(m+1)}\times n(m+1)} = \frac{1}{\mathrm{n(m+1)}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (16)$$

and H is the operational matrix of integration and can be obtained as:

$$\mathrm{H}_{\mathrm{n(m+1)}\times n(m+1)} = \frac{1}{2\mathrm{n(m+1)}} \begin{bmatrix} \frac{1}{35} & \frac{263}{105} & \frac{263}{105} & \frac{71}{35} \\ \frac{-3}{35} & \frac{17}{35} & \frac{87}{35} & \frac{67}{35} \\ \frac{3}{35} & \frac{-17}{35} & \frac{53}{35} & \frac{73}{35} \\ \frac{-1}{35} & \frac{17}{105} & \frac{-53}{105} & \frac{69}{35} \end{bmatrix} \quad (17)$$

The integration of the cross product of two OBH function vectors can be obtained as

$$D = \int_0^1 OBH_{(n\times(m+1))}(t)\, OBH^T_{(n\times(m+1))}(t)\ d(t) \tag{18}$$

$$\approx \begin{bmatrix} \mathrm{L} & 0 & \cdots & 0 \\ 0 & L & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & L \end{bmatrix}$$

where L is an $M \times (n+1)$ diagonal matrix given by

$$\mathrm{L} = \frac{1}{\mathrm{M}(n+\mathrm{M})} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{5} & \frac{1}{20} \\ \frac{1}{2} & \frac{3}{5} & \frac{9}{20} & \frac{1}{5} \\ \frac{1}{5} & \frac{9}{20} & \frac{3}{5} & \frac{1}{2} \\ \frac{1}{20} & \frac{1}{5} & \frac{1}{2} & 1 \end{bmatrix} \tag{19}$$

Eq. (14-18) are very important for solving Volterra- Fredholm integral equation of the second kind problems, because the D and P matrix can increase the calculating speed, as well as save the memory storage.

D. Multiplication of hybrid functions

It is usually necessary to evaluate $OBH_{(n\times(m+1))}(t)\ OBH^T_{(n\times(m+1))}(t)$ for the Volterra- Fredholm integral equation of the second kind via OBH functions:

Let the product of $OBH_{(n\times(m+1))}(t)$ and $OBH^T_{(n\times(m+1))}(t)$ be called the product matrix of OBH functions:

$$OBH_{(n(m+1))}(t) OBH^T_{(n(m+1))}(t) \cong M_{(n(m+1)\times n(m+1))}(t) \tag{20}$$

$$M_{(M(n+1)\times M(n+1))}(t) = \begin{bmatrix} OBH_{10}(t)OBH_{10}(t) & OBH_{10}(t)OBH_{20}(t) & \cdots & OBH_{10}(t)OBH_{M,n+1}(t) \\ OBH_{20}(t)OBH_{10}(t) & OBH_{20}(t)OBH_{20}(t) & \cdots & OBH_{20}(t)OBH_{M,n+1}(t) \\ OBH_{30}(t)OBH_{10}(t) & OBH_{30}(t)OBH_{20}(t) & \cdots & OBH_{30}(t)OBH_{M,n+1}(t) \\ \vdots & \vdots & \cdots & \vdots \\ OBH_{M,n+1}(t)OBH_{10}(t) & OBH_{M,n+1}(t)OBH_{20}(t) & \cdots & OBH_{M,n+1}(t)OBH_{M,n+1}(t) \end{bmatrix}$$

With the above recursive formulae, we can evaluate $M_{((M,n+1)\times M,n+1))}(t)$ for any M and n.

The matrix $M_{((M,n+1)\times_{M,n+1}))}(t)$ in (20) satisfies

$$M_{(M(n+1))}(t)\,c_{(M(n+1))} = C_{(M(n+1)\times M(n+1))}\,OBH_{(M(n+1))}(t) \tag{21}$$

where $c_{(n(m+1))}$ is defined in Eq. (10) and $C_{(n(m+1)\times n(m+1))}$ is called the coefficient matrix. We consider that $M=4$ and $n=3$. That is

$$M_{(16)\times 16)}(t)=\begin{bmatrix} OBH_{10}(t)\,OBH_{10}(t) & OBH_{10}(t)OBH_{20}(t) & \cdots & OBH_{10}(t)\,OBH_{44}(t) \\ OBH_{20}(t)\,OBH_{10}(t) & OBH_{20}(t)OBH_{20}(t) & \cdots & OBH_{20}(t)\,OBH_{44}(t) \\ OBH_{30}(t)\,OBH_{10}(t) & OBH_{30}(t)OBH_{20}(t) & \cdots & OBH_{30}(t)\,OBH_{44}\ (t) \\ \vdots & \vdots & \cdots & \vdots \\ OBH_{44}(t)\,OBH_{10}(t) & OBH_{44}(t)\,OBH_{20}(t) & \cdots & OBH_{441}(t)\,OBH_{441}(t) \end{bmatrix}$$

$$c_{(16)} \equiv [c_{10}, c_{20}, \cdots, c_{40}, c_{11}, c_{21}, \cdots, c_{41}, c_{12}, c_{22}, \cdots, c_{42}, c_{31}, c_{32}, \cdots, c_{43}] \tag{22}$$

and

$$\begin{aligned} OBH_{(16)}(t) \equiv [&OBH_{10}(t), OBH_{20}(t), \cdots, OBH_{40}(t), OBH_{11}(t), OBH_{21}(t), \cdots, \\ &OBH_{41}(t), OBH_{12}(t), OBH_{22}(t), \cdots, OBH_{42}(t), OBH_{31}(t), OBH_{32}(t), \cdots, OBH_{43}(t)]^T \end{aligned}$$

Using the vector $c_{(16)}$ in Eq. (22), the coefficient matrix $C_{16\times 16}$ in Eq. (21) determined by

$$C_{(M(n+1))\times_{(M(n+1))}} = \begin{bmatrix} C_0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & C_3 \end{bmatrix} \tag{23}$$

where $C_i\ \ , i=0,1,2,3$ are 4×4 matrices given by

$$
C_{i(M\times(n+1))}=\begin{bmatrix}
\frac{13}{4}c_{1i}+\frac{2}{21}c_{2i}-\frac{2}{105}c_{3i}-\frac{1}{210}c_{4i} & \frac{23}{24}c_{1i}+\frac{5}{14}c_{2i}+\frac{6}{35}c_{3i}+\frac{2}{105}c_{4i} & \frac{5}{21}c_{1i}+\frac{3}{14}c_{2i}-\frac{3}{70}c_{3i}+\frac{2}{105}c_{4i} & \frac{-1}{21}c_{1i}+\frac{1}{21}c_{2i}+\frac{1}{210}c_{3i}-\frac{1}{210}c_{4i}\\
\frac{13}{4}c_{1i}+\frac{2}{21}c_{2i}-\frac{2}{105}c_{3i}-\frac{1}{210}c_{4i} & \frac{23}{24}c_{1i}+\frac{5}{14}c_{2i}+\frac{6}{35}c_{3i}+\frac{2}{105}c_{4i} & \frac{5}{21}c_{1i}+\frac{3}{14}c_{2i}-\frac{3}{70}c_{3i}+\frac{2}{105}c_{4i} & \frac{-1}{21}c_{1i}+\frac{1}{21}c_{2i}+\frac{1}{210}c_{3i}-\frac{1}{210}c_{4i}\\
\frac{13}{4}c_{1i}+\frac{2}{21}c_{2i}-\frac{2}{105}c_{3i}-\frac{1}{210}c_{4i} & \frac{23}{24}c_{1i}+\frac{5}{14}c_{2i}+\frac{6}{35}c_{3i}+\frac{2}{105}c_{4i} & \frac{5}{21}c_{1i}+\frac{3}{14}c_{2i}-\frac{3}{70}c_{3i}+\frac{2}{105}c_{4i} & \frac{-1}{21}c_{1i}+\frac{1}{21}c_{2i}+\frac{1}{210}c_{3i}-\frac{1}{210}c_{4i}\\
\frac{13}{4}c_{1i}+\frac{2}{21}c_{2i}-\frac{2}{105}c_{3i}-\frac{1}{210}c_{4i} & \frac{23}{24}c_{1i}+\frac{5}{14}c_{2i}+\frac{6}{35}c_{3i}+\frac{2}{105}c_{4i} & \frac{5}{21}c_{1i}+\frac{3}{14}c_{2i}-\frac{3}{70}c_{3i}+\frac{2}{105}c_{4i} & \frac{-1}{21}c_{1i}+\frac{1}{21}c_{2i}+\frac{1}{210}c_{3i}-\frac{1}{210}c_{4i}
\end{bmatrix}
$$

With the powerful properties of Eqs. (13-23), the solution of Volterra-Fredholm integral equation of the second kind can be easily found.

4. Solution of Volterra- Fredholm integral equation of the second kind via hybrid functions

Consider the following integral equation:

$$y(x)=f(x)+\int_0^1 k_1(x,t)\,y(t)\,dt+\int_0^x k_2(x,t)\,y(t)\,dt \tag{24}$$

$$y(x)\approx Y^T OBH(x)$$

$$k_1(x,t)\approx OBH^T(x)\,K_1\,OBH(t)$$

$$k_2(x,t)\approx OBH^T(x)\,K_2\,OBH(t)$$

$$f(x)\approx F^T OBH(x)$$

with substituting in Eq. (24)

$$
\begin{aligned}
OBH^T(x)Y &= OBH^T(x)F+\int_0^1 OBH^T(x)\,K_1\,OBH(t)\;OBH^T(t)Y\,dt\\
&\quad+\int_0^x OBH^T(x)\,K_2\,OBH(t)\,OBH^T(t)Y\,dt
\end{aligned} \tag{25}
$$

$$OBH^T(x)Y = OBH^T(x)F + OBH^T(x)K_1 \int_0^1 OBH(t)\ OBH^T(t)Y\,dt$$
$$+ OBH^T(x)K_2 \int_0^x OBH(t)\,OBH^T(t)Y\,dt$$

Applying Eqs. (10), (12) and (20) to Eq. (25) and Eq.(25) becomes

$$OBH^T(x)Y = OBH^T(x)F + OBH^T(x)K_1\ DY + OBH^T(x)K_2 \int_0^x \tilde{Y}\,OBH(t)\,dt \tag{26}$$

where $\tilde{Y}\,OBH(t) = M(t)Y = OBH(t)\,OBH^T(t)Y$ is a copy of (21). The integrals of (26) can be obtained by multiplying the operation matrix of integration of (14) as follows:

$$OBH^T(x)Y = OBH^T(x)F + OBH^T(x)K_1\ DY + OBH^T(x)K_2\ \tilde{Y}\ P\,OBH(x) \tag{27}$$

In order to find Y we collocate Eq. (27) in $M(n+1)$ nodal points of Newton-Cotes [9] as

$$t_i = \frac{2i-1}{2M(n+1)} \tag{28}$$

From Eqs. (27) and (28), we have a system of $M(n+1)$ linear equations and $M(n+1)$ unknowns. After solving above linear system, we can achieve the unknown vectors Y. The required approximated solution $y(x)$ for Volterra–Fredholm integral Eq. (1) can be obtained by using Eqs.(22), (26) and (27) as follows

$$y(x) = f(x) + OBH^T(x)K_1\ DY + OBH^T(x)K_2\ \tilde{Y}\ P\,OBH(x)$$

5. Numerical Examples

We applied the presented schemes to the following Volterra- Fredholm integral equation of second kind. For this purpose, we consider two examples.

Example 1: Consider the following linear Volterra- Fredholm integral equation [28].

$$\begin{gathered} y(x) = f(x) + \int_0^1 xt\ y(t)\,dt + \int_0^x xt\ y(t)\,dt \\ f(x) = \frac{2}{3}x - \frac{1}{3}x^4 \end{gathered} \tag{29}$$

If we solve (29) for $y(x)$ directly, the analytic solution can be shown to be $y(x) = x$.

The comparison among the OBH solution and the analytic solution for $t \in [0,1)$ is shown in Table 1 for M=4 and n=3, which confirms that the OBH method gives better solution as the

Scaling Function Interpolation method. The average relative errors of our method 6.12574987×10^{-6}. Better approximation is expected by choosing the optimal values of M and n.

Table.1. The comparison among OBH and Scaling Function Interpolation method for example 1

x	OBH solution	Analytic solution	Absolute errors of OBH method	Absolute errors of Scaling Function Interpolation method[28]
0.1	0.10000003	0.1	3×10^{-8}	3.348×10^{-7}
0.2	0.19999999	0.2	1×10^{-8}	1.263×10^{-7}
0.3	0.29999999	0.3	1×10^{-8}	1.905×10^{-7}
0.4	0.40000002	0.4	2×10^{-8}	2.564×10^{-8}
0.5	0.49999999	0.5	1×10^{-8}	1.316×10^{-8}
0.6	0.60000001	0.6	1×10^{-8}	1.876×10^{-7}
0.7	0.69999999	0.7	1×10^{-8}	6.735×10^{-7}
0.8	0.79999999	0.8	1×10^{-8}	2.064×10^{-7}
0.9	0.90000007	0.9	7×10^{-8}	2.589×10^{-7}

Example 2: Consider the following linear Volterra- Fredholm integral equation [29].

$$y(x)=f(x)+\int_0^x (x^2-t)\,y(t)\,dt+\int_0^1 (xt+x)\;y(t)\,dt \qquad (30)$$
$$f(x)=e^x+e^x x-e^x-xe-x^2e^x+x^2+1$$

With the exact solution $y(x)=e^x$

The comparison among the OBH solution and the analytic solution for $t\in[0,1)$ is shown in Table 2 for M=2 and n=1 which confirms that the OBH method gives almost the same solution as the analytic method. The average relative errors of our method 7.64518×10^{-8} at M=8, n=7 . Better approximation is expected by choosing the higher values of M and n.

Table.2. The comparison among OBH and analytic solutions for **example 2**

x	OBH solution	The Exact Solution	Absolute errors of OBH method at M=4,n=3	Absolute errors of OBH method at M=8 , n=7
0.1	1.105134	1.10586745	7.3345×10^{-4}	4.532×10^{-9}
0.2	1.221474	1.2217852	3.112×10^{-4}	3.156×10^{-8}
0.3	1.349841	1.349112	7.29×10^{-4}	9.653×10^{-7}
0.4	1.491835	1.491474	3.61×10^{-4}	7.261×10^{-8}
0.5	1.648742	1.648536	2.06×10^{-4}	8.146×10^{-8}
0.6	1.822146	1.822787	6.41×10^{-4}	5.745×10^{-7}
0.7	2.013712	2.013752707	4.0707×10^{-5}	3.541×10^{-6}
0.8	2.2255464	2.225540928	5.472×10^{-6}	2.521×10^{-7}
0.9	2.45960213	2.459603111	9.81×10^{-7}	3.348×10^{-6}

6. Conclusion

In this paper, we have worked out a combination of orthonormal Bernstein and Block-Pulse functions to approximating solution of linear Volterra- Fredholm integral equations. The method is based upon reducing the system into a set of algebraic equations. The generation of this system needs just sampling of functions multiplication and addition of matrices and needs no integration. The matrix D and P are sparse; hence are much faster than other functions and reduces the CPU time and the computer memory, at the same time keeping the accuracy of the solution. The numerical examples support this claim. Also we noted that when the degree of Hybrid Orthonormal Bernstein and Block-Pulse Functions is increasing the errors decreasing to smaller values. The results show that the proposed method is a promising tool for this type of linear Volterra- Fredholm integral equations. The main advantage of these methods are the ability , reliability and low cost of setting up the equations without using any projection method.

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